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Unitarity and the Isobar Model: Two-Body Discontinuities*

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In the isobar model the fact that the final-state isobar channels overlap calls for an investigation in which unitarity provides constraints on the construction. The discontinuities in all the physical two-body subenergies are treated, allowing for two-body intermediate states. The process $KN \rightarrow K\pi N$ is used for illustration throughout. The discontinuity formulas are applied to the isobar expansion of the amplitude; rather explicit attention is given to the details of spin in order to show how the recoupling problems can be unraveled. The resulting subenergy discontinuities take the form of integrations across the Dalitz plot. An example is given in conclusion in which s-waves dominate in the final state, so that a modest number of coupled isobar amplitudes enter in the constraint relations.

1. INTRODUCTION

Hadron reactions leading to three-body final states have continued to present considerable analytical complexity. Their analyses have great practical significance because they offer a means of obtaining two-body information about systems to which we have almost no other access. Phenomenologies relating to the $\pi\pi$ interaction in $\pi N \rightarrow \pi\pi N$ and the $K\pi$ interaction in $KN \rightarrow K\pi N$ are prime examples. Our interest in such production processes therefore calls for their construction in terms of two-body interactions. The isobar model has long been adopted as the procedure for doing this.

In the isobar model, separate channels are associated with the three possible ways of organizing the three-body final state as a two-body isobar plus a third particle. The model then expresses the two-body to three-body production amplitude as the sum of the amplitudes to the separate channels. Each term contributing to the production process takes the form of an amplitude leading to three particles, two of which are in a state with definite quantum numbers (an isobar state). This construction in terms of isobar amplitudes is indicated in Fig. 1. It is clear that overlapping of states occurs in this scheme and it would appear that opportunities for multiple-counting effects arise. To suppress these effects the constraints due to unitarity must be invoked. The implications of these constraints have been investigated by Aaron and Amado [1]

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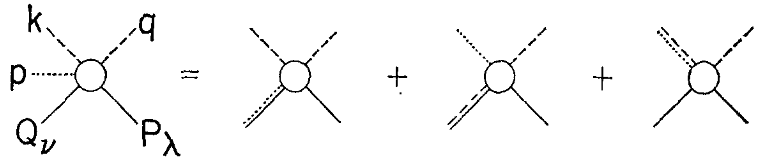

FIG. 1. The production amplitude as a sum of isobar amplitudes.

and by Aitchison [2]. Their studies pertain to the use of unitarity in the pair-subenergy variables of the production amplitude; the treatments in Refs. [1, 2] are of systems of particles without spin.

Our phenomenological motives, cited in the first paragraph, concern processes with a nucleon target. This work thus has to deal with all the details due to spin and isospin. For definiteness we choose to develop the formalism for $KN \to K\pi N$, an example for which the notation can be made reasonably transparent.

The momenta and helicities are as shown in Fig. 1, in which

$$K(q) + N(P_\lambda) \to K(k) + \pi(p) + N(Q_\nu). \tag{1}$$

The invariant mass variables, which are physical for the production process, are

$$\begin{aligned} &\text{the final } \pi N \text{ mass}^2 && w_1^2 = -(Q+p)^2, \\ &\text{the final } KN \text{ mass}^2 && w_2^2 = -(Q+k)^2, \\ &\text{the final } K\pi \text{ mass}^2 && x = -(k+p)^2, \\ &\text{and the total mass}^2 && W^2 = -(P+q)^2. \end{aligned} \tag{2}$$

These variables are related by

$$w_1^2 + w_2^2 + x = W^2 + M^2 + m^2 + \mu^2, \tag{3}$$

where M, m, and μ are the masses of the N, K, and π.

It is useful to express the amplitude for reaction (1) in the three equivalent forms

$$N(QpPq)\langle Qp \text{ out} \mid j_K \mid Pq \text{ in}\rangle, \tag{4}$$

$$N(QkPq)\langle Qk \text{ out} \mid j_\pi \mid Pq \text{ in}\rangle, \tag{5}$$

and

$$N(kpPq)\, \bar{u}_Q \langle kp \text{ out} \mid f \mid Pq \text{ in}\rangle. \tag{6}$$

The factor $N(Qp \ldots)$ contains the normalization of states; e.g., $N(Q) = (Q_0/M)^{1/2}$, $N(p) = (2p_0)^{1/2}$, etc. The discontinuity in the variable w_1 may then be expressed in terms of (4) as

$$N(QpPq)(\langle Qp \text{ out} \mid j_K \mid Pq \text{ in}\rangle - \langle Qp \text{ in} \mid j_K \mid Pq \text{ in}\rangle), \tag{7}$$

in which the first and second terms are related by analytic continuation from $(W + i0, w_1 + i0)$ to $(W + i0, w_1 - i0)$. By means of standard methods, expression (7) becomes

$$i(2\pi)^4 \textstyle\sum'' \delta(Q'' + p'' - Q - p)\, N(QPq) \langle Q \mid j_\pi \mid Q''p'' \text{ out}\rangle\langle Q''p'' \text{ out} \mid j_K \mid Pq \text{ in}\rangle. \tag{8}$$

The summation is over all the degrees of freedom of the intermediate πN-state. The first matrix element in (8), when multiplied by the normalization factor $N(QQ''p'')$, is the elastic $\pi N \to \pi N$ amplitude, continued to $w_1 - i0$. If we proceed in a similar fashion, the discontinutity in w_2 may be found from (5) to be

$$i(2\pi)^4 \sum'' \delta(Q'' + k'' - Q - k)\, N(QPq)\, \langle Q \mid j_K \mid Q''k'' \text{ out}\rangle\langle Q''k'' \text{ out} \mid j_\pi \mid Pq \text{ in}\rangle, \quad (9)$$

in which there appears the $KN \to KN$ amplitude, continued to $w_2] - i0$. Likewise, the discontinuity in x may be derived from (6):

$$i(2\pi)^4 \sum'' \delta(k'' + p'' - k - p)\, N(kPq)\, \bar{u}_Q\langle k \mid j_\pi \mid k''p'' \text{ out}\rangle\langle k''p'' \text{ out} \mid f \mid Pq \text{ in}\rangle, \quad (10)$$

wherein there occurs the $K\pi \to K\pi$ amplitude, continued to $x - i0$. These discontinuity relations are shown pictorially in Fig. 2. Also shown in Fig. 2 is the result for the discontinuity in W:

$$i(2\pi)^4 \sum'' \delta(P'' + q'' - P - q)\, N(QpP)\, \langle Qp \text{ out} \mid j_K \mid P''q'' \text{ in}\rangle\langle P''q'' \text{ in} \mid j_K \mid P\rangle, \quad (11)$$

in which the $KN \to KN$ amplitude appears, continued to $W - i0$.

Of course it should be emphasized that only the two-particle contributions have been given in the foregoing unitarity relations. Expressions (8), (9) and (10) are complete only if the subenergies are less than their respective inelastic thresholds. The discontinuity in W, result (11), is notably incomplete without the contribution from

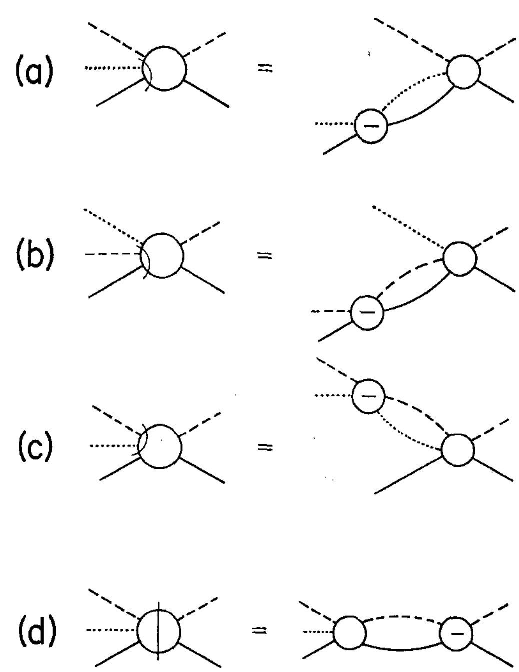

FIG. 2. Discontinuities of the production amplitude: (a) in the πN subenergy w_1, (b) in the KN subenergy w_2, (c) in the $K\pi$ subenergy[2] x, and (d) in the total energy W.

the $K\pi N$ intermediate state. In this work we restrict our attention to the effects of two-body unitarity in all the physical subenergy channels, and show how these effects constrain the isobar model. It is evident that the subenergy discontinuitites of the isobar amplitudes are constraints which lead to integral equations by means of dispersion relations. It will be helpful to visualize what follows in terms of Figs. 1 and 2; we take the isobar expansion of Fig. 1, feed it into the results of Fig. 2, and extract the consequences. This construction is illustrated in Figs. 3, 4, and 5. It should be noted that the evaluation of a given subenergy discontinuity singles out only that isobar configuration which has that subenergy for one of its variables. On the right-hand side in each of these three figures there are three contributions, two of which involve the participation of the "other" two isobars. Our chief objective is to show how these recoupling contributions are unraveled.

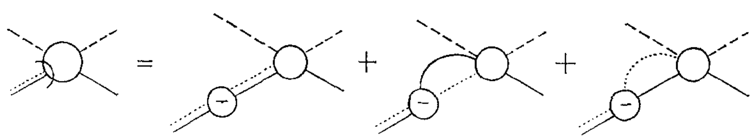

FIG. 3. The w_1-discontinuity in the isobar model.

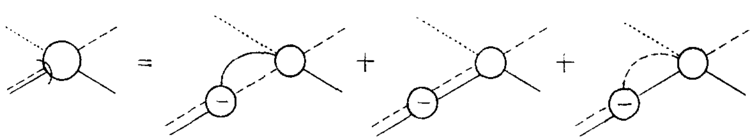

FIG. 4. The w_2-discontinuity in the isobar model.

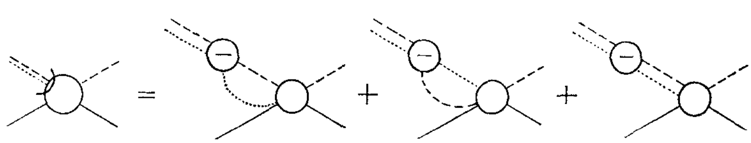

FIG. 5. The x-discontinuity in the isobar model.

2. ISOSPIN

The role of isospin in this investigation is conveniently handled in terms of projection operators. We let the π have a Cartesian isovector index i, and represent the K and the N by isospinors. The elastic amplitudes may then be expanded in amplitudes of definite isospin:

$$\begin{aligned} M(\pi_i N \to \pi_j N) &= \sum_{t_1} a_{ji}^{t_1} M^{t_1}, \\ M(KN \to KN) &= \sum_{t_2} b^{t_2} M^{t_2}, \qquad (12) \\ M(K\pi_i \to K\pi_j) &= \sum c_{ji}^{t} M^{t}. \end{aligned}$$

The three contributions to the production amplitude shown in Fig. 1 may be expanded in amplitudes of definite total isospin T and isobar isospin:

$$\begin{aligned} M(KN \to (\pi_i N)\, K) &= \sum_{Tt_1} \mathcal{A}_i^{Tt_1} M^{Tt_1}, \\ M(KN \to (KN)\, \pi_i) &= \sum_{Tt_2} \mathcal{B}_i^{Tt_2} M^{Tt_2}, \\ M(KN \to (K\pi_i)\, N) &= \sum_{Tt} \mathcal{C}_i^{Tt} M^{Tt}. \end{aligned} \tag{13}$$

We have listed the projection operators a, b, c, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in Appendix A.

The unitarity relation shown in Fig. 3 reads

$$\begin{aligned} \mathrm{disc}_{w_1} \sum_{Tt_1} \mathcal{A}_j^{Tt_1} M^{Tt_1} \\ = 2\pi i \sum_{w_1 i} \left(\sum_{t_1'} a_{ji}^{t_1'} M_-^{t_1'} \right) \left(\sum_{Tt_1} \mathcal{A}_i^{Tt_1} M^{Tt_1} + \sum_{Tt_2} \mathcal{B}_i^{Tt_2} M^{Tt_2} + \sum_{Tt} \mathcal{C}_i^{Tt} M^{Tt} \right). \end{aligned} \tag{14}$$

The symbol $\sum_{w_1}$ refers to the summation over all the intermediate-state phase space in the w_1-channel. Here, and throughout, the subscript $(-)$ denotes analytic continuation to the bottom of the relevant cut, in this case to $w_1 - i0$. When we use formulas (A7) we obtain the result

$$\mathrm{disc}_{w_1} M^{Tt_1} = 2\pi i \sum_{w_1} M_-^{t_1} \left(M^{Tt_1} + \sum_{t_2} C_{t_1 t_2}^T M^{Tt_2} + \sum_t C_{t_1 t}^T M^{Tt} \right). \tag{15}$$

The C's are constants, elements of the crossing matrices, recorded in Table I. The relation shown in Fig. 4 reads

$$\begin{aligned} \mathrm{disc}_{w_2} \sum_{Tt_2} \mathcal{B}_j^{Tt_2} M^{Tt_2} \\ = 2\pi i \sum_{w_2} \left(\sum_{t_2'} b^{t_2'} M_-^{t_2'} \right) \left(\sum_{Tt_1} \mathcal{A}_j^{Tt_1} M^{Tt_1} + \sum_{Tt_2} \mathcal{B}_j^{Tt_2} M^{Tt_2} + \sum_{Tt} \mathcal{C}_j^{Tt} M^{Tt} \right). \end{aligned} \tag{16}$$

We obtain, with the help of Eqs. (A8),

$$\mathrm{disc}_{w_2} M^{Tt_2} = 2\pi i \sum_{w_2} M_-^{t_2} \left(\sum_{t_1} C_{t_2 t_1}^T M^{Tt_1} + M^{Tt_2} + \sum_t C_{t_2 t}^T M^{Tt} \right). \tag{17}$$

In the same fashion, the relation shown in Fig. 5, which reads

$$\begin{aligned} \mathrm{disc}_x \sum_{Tt} \mathcal{C}_j^{Tt} M^{Tt} \\ = 2\pi i \sum_{xi} \left(\sum_{t'} c_{ji}^{t'} M_-^{t'} \right) \left(\sum_{Tt_1} \mathcal{A}_i^{Tt_1} M^{Tt_1} + \sum_{Tt_2} \mathcal{B}_i^{Tt_2} M^{Tt_2} + \sum_{Tt} \mathcal{C}_i^{Tt} M^{Tt} \right), \end{aligned} \tag{18}$$

TABLE I

Crossing-Matrix Elements

$(C^T_{tt_1})$		$T = 0$			$T = 1$	
		$\frac{1}{2}$	$\frac{3}{2}$		$\frac{1}{2}$	$\frac{3}{2}$
	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}\, 2^{1/2}$
	$\frac{3}{2}$	0	0	$\frac{3}{2}$	$\frac{2}{3}\, 2^{1/2}$	$-\frac{1}{3}$
$(C^T_{tt_2})$		$T = 0$			$T = 1$	
		0	1		0	1
	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{3}\, 3^{1/2}$	$-\frac{1}{3}\, 6^{1/2}$
	$\frac{3}{2}$	0	0	$\frac{3}{2}$	$\frac{1}{3}\, 6^{1/2}$	$-\frac{1}{3}\, 3^{1/2}$
$(C^T_{t_1t_2})$		$T = 0$			$T = 1$	
		0	1		0	1
	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{3}\, 3^{1/2}$	$-\frac{1}{3}\, 6^{1/2}$
	$\frac{3}{2}$	0	0	$\frac{3}{2}$	$-\frac{1}{3}\, 6^{1/2}$	$-\frac{1}{3}\, 3^{1/2}$

becomes, with the aid of Eqs. (A9)

$$\operatorname{disc}_x M^{Tt} = 2\pi i \sum_x M_-{}^t \Big(\sum_{t_1} C^T_{tt_1} M^{Tt_1} + \sum_{t_2} C^T_{tt_2} M^{Tt_2} + M^{Tt} \Big). \tag{19}$$

All crossing-matrix elements appearing in (15), (17), and (19) have been listed in Table I.

It is obvious that the evaluation of the W-discontinuity, expression (11) and Fig. 2d, leads to the results

$$\operatorname{disc}_W M^{Tt_1} = 2\pi i \sum_W M^{Tt_1} M_-{}^T,$$

similarly for $1 \to 2$, and (20)

$$\operatorname{disc}_W M^{Tt} = 2\pi i \sum_W M^{Tt} M_-{}^T.$$

In Eqs. (20) the factor $M_-{}^T$ stands for the $KN \to KN$ amplitude, with isospin $T = 0$ and 1, continued to $W - i0$.

3. Isobar Expansion

The amplitudes in the isobar expansion of Fig. 1 correspond to terms in an angular momentum decomposition. Summation is implied over quantum numbers associated with the total angular momentum, JM, and the isobar angular momenta, $j_1 m_1$ for

πN, $j_2 m_2$ for KN, and lm for $K\pi$. There also occur assorted helicities, combinations of which correspond to states of definite parity. We follow the development due to Wick [3], and are particular about our definitions so as to minimize the number of troublesome phases which can enter the construction.

To describe momenta and helicities $P_\lambda + q \rightarrow Q_\nu + k + p$ in the overall center-of-mass system (CM) we adopt the reference configurations shown in Fig. 6. For

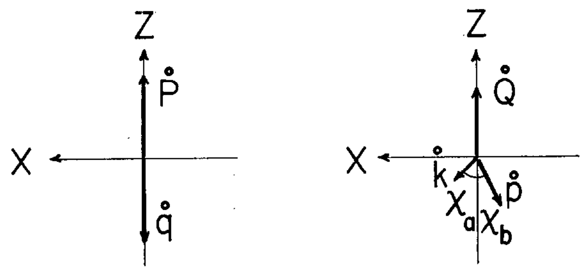

FIG. 6. Reference configurations for CM momenta in the initial and final states. The Y axis points toward the reader.

the initial state the nucleon momentum $\mathring{P}$ is along the Z axis; for the final state the momenta $\mathring{Q}$, $\mathring{k}$, and $\mathring{p}$ lie in the XZ plane with the nucleon momentum $\mathring{Q}$ along the Z axis. Rotations and Lorentz transformations then generate the states for which we desire angular momentum expansions. Even though some rotation angles can be chosen to vanish we let them all be arbitrary; the bookkeeping is no more cumbersome and in fact is more symmetrical this way.

If we identify the rotation r whose Euler angles are $(\alpha\beta 0)$ such that $P = r\mathring{P}$ then the initial state in CM is expanded as

$$| P_\lambda q\rangle = R \,| \mathring{P}_\lambda \mathring{q}\rangle = \sum_{JM} N_J D^J_{M\lambda}(r) \,| P(W)\, JM\lambda\rangle. \tag{21}$$

Throughout, we use the notation $N_J = ((2J + 1)/4\pi)^{1/2}$.

In the final state we define the three vectors in CM:

$$\begin{aligned} Q_a &= Q + p, \\ Q_b &= Q + k, \\ K_c &= k + p. \end{aligned} \tag{22}$$

Three different angular momentum coupling schemes are possible and all three are needed. Suitable rotations of $\mathring{Q}\mathring{k}\mathring{p}$ take us to states for which we can use the basic formula [3, Eq. (24)] and its inverse, or a variant of it.

The Y-rotation, $r_{0\chi_a 0}$, applied to the reference configuration rotates $\mathring{k}$ to the negative Z axis. In the rest frame of Q_a (CM$_1$), the nucleon momentum is Q_1 with direction

(ϑ_1, ψ_1). If the subsequent rotation r_1, whose Euler angles are $(\phi_1\theta_1\psi_1)$, leads to the final state in CM, then we can expand the final state as

$$\begin{aligned} | Q_\nu kp\rangle &= R_1 R_{0\chi_a 0} | \mathring{Q}_\nu \mathring{k}\mathring{p}\rangle \\ &= \sum_{\substack{JM \\ j_1 m_1 \mu}} N_J N_{j_1} d^{1/2}_{\nu\mu}(\omega_1)\, d^{j_1}_{m_1\mu}(\vartheta_1)\, D^J_{Mm_1}(r_1) \,|\, Q_a(W)\, JM,\, Q_1(w_1)\, j_1 m_1 \mu\rangle. \end{aligned} \tag{23}$$

A nucleon spin rotation matrix enters; its angle ω_1 is deduced from the Lorentz transformation from CM_1 and is given later.

The Y-rotation, $r^{-1}_{0\chi_b 0}$, followed by the Z-rotation, $r_{00\pi}$, applied to $\mathring{Q}\mathring{k}\mathring{p}$ rotates $\mathring{p}$ to the negative Z axis; the Z-rotation obviates the need for a phase factor in the application of Wick's basic formula because it causes $\mathring{Q}$ to be rotated to zero azimuthal position preparatory to rotation into its final orientation. In the rest frame of Q_b (CM_2), the nucleon momentum is Q_2 with direction (ϑ_2, ψ_2). If the subsequent rotation r_2, with Euler angles $(\phi_2\theta_2\psi_2)$, leads to the final CM state, then

$$\begin{aligned} | Q_\nu kp\rangle &= R_2 R_{00\pi} R^{-1}_{0\chi_b 0} | \mathring{Q}_\nu \mathring{k}\mathring{p}\rangle \\ &= \sum_{\substack{JM \\ j_2 m_2 \mu}} N_J N_{j_2} d^{1/2}_{\nu\mu}(\omega_2)\, d^{j_2}_{m_2\mu}(\vartheta_2)\, D^J_{Mm_2}(r_2) \,|\, Q_b(W)\, JM,\, Q_2(w_2)\, j_2 m_2 \mu\rangle. \end{aligned} \tag{24}$$

The nucleon spin rotation angle ω_2 is determined by the Lorentz transformation from CM_2 and, like ω_1, is given later.

The third coupling scheme proceeds directly from the reference configuration. If r_3, with Euler angles $(\phi_3\theta_3\psi_3)$, applied to $\mathring{Q}\mathring{k}\mathring{p}$, leads to the final CM state then the expansion of $| Q_\nu kp\rangle = R_3 | \mathring{Q}_\nu \mathring{k}\mathring{p}\rangle$ is desired. Wick's formula must be modified here because the isobar momentum, $\mathring{K}_c = \mathring{k} + \mathring{p}$, is oriented down, not up, along the Z axis. The resulting expansion is transparent enough, however:

$$\begin{aligned} | Q_\nu kp\rangle &= R_3 | \mathring{Q}_\nu \mathring{k}\mathring{p}\rangle \\ &= \sum_{\substack{JM \\ lm}} N_J N_l d^l_{-m0}(\vartheta_3)\, D^J_{M,\nu-m}(r_3) \,|\, Q(W)\, JM,\, k_3(x)\, lm,\, \nu\rangle. \end{aligned} \tag{25}$$

In the isobar rest frame in which K_c has been transformed to rest (CM_3), the K meson has momentum k_3 with direction (ϑ_3, ψ_3).

In this bewildering array of definitions we note that in CM, Q_a (the πN isobar) has direction (θ_1, ϕ_1), Q_b (the KN isobar) has direction (θ_2, ϕ_2), and K_c (the $K\pi$ isobar) has direction $(\pi - \theta_3, \phi_3 - \pi)$. Figure 7 provides a summary of all the angles we have defined. The various rotations applied to the reference configuration are connected to each other by the important relation

$$r_1 r_{0\chi_a 0} = r_2 r_{00\pi} r^{-1}_{0\chi_b 0} = r_3 . \tag{26}$$

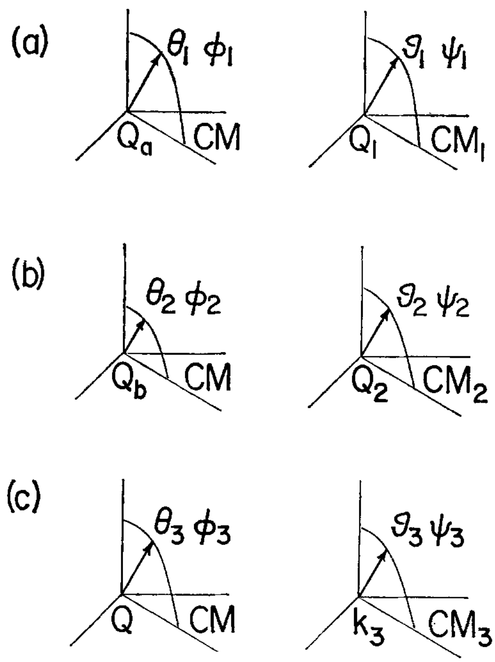

FIG. 7. Angles defined for the three different coupling schemes in the final state: (a) applies to Eq. (23), (b) to Eq. (24), and (c) to Eq. (25).

The isobar expansion follows from these angular momentum decompositions. The amplitudes of definite isospin T and definite isobar isospin introduced in (13) are expanded as

$$M^{Tt_1} = \sum_{\substack{JM \\ j_1 m_1 \mu}} N_J{}^2 N_{j_1} d^{1/2}_{\nu\mu}(\omega_1)\, d^{j_1}_{m_1\mu}(\vartheta_1)\, D^{J*}_{Mm_1}(r_1)\, \langle m_1\mu \mid M^{Jj_1} \mid \lambda\rangle^{Tt_1} D^J_{M\lambda}(r), \tag{27}$$

$$M^{Tt_2} = \sum_{\substack{JM \\ j_2 m_2 \mu}} N_J{}^2 N_{j_2} d^{1/2}_{\nu\mu}(\omega_2)\, d^{j_2}_{m_2\mu}(\vartheta_2)\, D^{J*}_{Mm_2}(r_2)\, \langle m_2\mu \mid M^{Jj_2} \mid \lambda\rangle^{Tt_2} D^J_{M\lambda}(r), \tag{28}$$

$$M^{Tt} = \sum_{\substack{JM \\ lm}} N_J{}^2 N_l d^l_{-m0}(\vartheta_3)\, D^{J*}_{M,\nu-m}(r_3)\, \langle \nu m \mid M^{Jl} \mid \lambda\rangle^{Tt} D^J_{M\lambda}(r). \tag{29}$$

The isobar amplitudes on the right-hand sides of these equations have definite J and definite isobar spins. In addition these quantities depend on isobar and initial nucleon helicities; the parent final nucleon helicity appears in (27) and (28), while the outgoing nucleon helicity appears in (29). We defer the formation of amplitudes having definite isobar parity and definite overall parity until later. The isospin indices on which these amplitudes also depend are suppressed in the manipulations to follow.

4. Subenergy Discontinuities

The w_1-discontinuity is expressed in Eq. (15) and in Fig. 3. On the left-hand side we use Eq. (27). On the right-hand side we use Eqs. (27), (28), and (29) with intermediate-state variables $(\vartheta_1'' r_1'')$, $(\vartheta_2'' r_2'')$, and $(\vartheta_3'' r_3'')$, respectively, corresponding to the πN variables $Q_{\nu''}'' p''$ over which we must sum. In addition we also need the πN elastic amplitude, expressed in CM so that spin rotations of the intermediate and final nucleon are required. When we assemble all these pieces we have

$$
\begin{aligned}
\operatorname{disc}_{w_1} & \sum_{\substack{JM \\ j_1 m_1 \mu}} N_J{}^2 N_{j_1} d^{1/2}_{\nu\mu}(\omega_1)\, d^{j_1}_{m_1\mu}(\vartheta_1)\, D^{J*}_{Mm_1}(r_1) \langle m_1\mu \mid M^{Jj_1} \mid \lambda\rangle D^J_{M\lambda}(r) \\
&= i(2\pi)^4 \sum_{\nu''} \int \frac{d^3Q''\, d^3p''}{(2\pi)^6\, Q_0'' p_0''} \frac{M}{2}\, \delta(Q'' + p'' - Q - p) \\
&\times \sum_{j_1' m_1' \nu' \lambda'} N^2_{j_1'} d^{1/2}_{\nu\nu'}(\omega_1)\, D^{j_1'*}_{m_1'\nu'}(\hat{r}_1) \langle \nu' \mid M^{j_1'} \mid \lambda'\rangle_-\, D^{j_1'}_{m_1'\lambda'}(\hat{r}_1'')\, d^{1/2}_{\nu''\lambda'}(\omega_1'') \\
&\times \sum_{JM} N_J{}^2 \Bigg\{ \sum_{j_1 m_1 \mu} N_{j_1} d^{1/2}_{\nu''\mu}(\omega_1'')\, d^{j_1}_{m_1\mu}(\vartheta_1'')\, D^{J*}_{Mm_1}(r_1'') \langle m_1\mu \mid M^{Jj_1} \mid \lambda\rangle \\
&+ \sum_{j_2 m_2 \mu} N_{j_2} d^{1/2}_{\nu''\mu}(\omega_2'')\, d^{j_2}_{m_2\mu}(\vartheta_2'')\, D^{J*}_{Mm_2}(r_2'') \langle m_2\mu \mid M^{Jj_2}_{\mathrm{C}} \mid \lambda\rangle \\
&+ \sum_{lm} N_l d^l_{-m0}(\vartheta_3'')\, D^{J*}_{M,\nu''-m}(r_3'') \langle \nu'' m \mid M^{Jl}_{\mathrm{C}} \mid \lambda\rangle \Bigg\} D^J_{M\lambda}(r).
\end{aligned}
\tag{30}
$$

Since isospin indices have been suppressed, the subscript C in the last two terms reminds us of the presence of the crossing matrices in Eq. (15). The Euler angles for the initial and final states have already been defined. In addition there are D-functions in (30) having the Euler angles $(\psi_1\vartheta_1 0)$ for $\hat{r}_1$ and $(\psi_1''\vartheta_1'' 0)$ for $\hat{r}_1''$, the respective directions of Q_1 and Q_1'' in CM_1. We note that r_1'' has Euler angles $(\phi_1\theta_1\psi_1'')$ because the momentum vector k is in a fixed direction while we integrate over Q'' and p''; it follows that

$$
D^{J*}_{Mm_1}(r_1'') = D^{J*}_{Mm_1}(r_1)\, e^{im_1(\psi_1'' - \psi_1)}. \tag{31}
$$

The integration is carried out in CM_1.

The contribution from the first term in braces on the right-hand side (the πN isobars) is

$$
2\pi i \rho_1 \sum_{\substack{JM \\ j_1 m_1 \mu \\ \lambda'}} N_J^2 N_{j_1} d^{1/2}_{\nu\mu}(\omega_1)\, d^{j_1}_{m_1\mu}(\vartheta_1)\, D^{J*}_{Mm_1}(r_1) \langle \mu \mid M^{j_1} \mid \lambda'\rangle_- \langle m_1\lambda' \mid M^{Jj_1} \mid \lambda\rangle D^J_{M\lambda}(r), \tag{32}
$$

in which the πN phase space factor is

$$
\rho_1(w_1) = MQ_1(w_1)/16\pi^3 w_1. \tag{33}
$$

The contributions from the KN isobars and the $K\pi$ isobars are more complicated. To perform the integrations over them we note, from Eq. (26), that

$$r_2'' = r_1'' r_{0(\chi_a''+\chi_b'')0} r_{00\pi}^{-1} \tag{34}$$

and

$$r_3'' = r_1'' r_{0\chi_a''0} . \tag{35}$$

From (34) it follows that

$$D^{J*}_{Mm_2}(r_2'') = \sum_n D^{J*}_{Mn}(r_1'')\, d^J_{nm_2}(\chi_a'' + \chi_b'')\, e^{-i\pi m_2}, \tag{36}$$

and from (35) that

$$D^{J*}_{M,\nu''-m}(r_3'') = \sum_n D^{J*}_{Mn}(r_1'')\, d^J_{n,\nu''-m}(\chi_a''). \tag{37}$$

We can then use Eq. (31) and do the ψ_1'' integration. The second and third terms in braces on the right-hand side of (30) become

$$\begin{aligned} &2\pi i\rho_1 \sum_{\substack{JM\\ j_1 m_1 \mu\\ \lambda'\nu''}} N_J{}^2 N_{j_1} d^{1/2}_{\nu\mu}(\omega_1)\, d^{j_1}_{m_1\mu}(\vartheta_1)\, D^{J*}_{Mm_1}(r_1)\, \langle \mu \mid M^{j_1} \mid \lambda' \rangle_- \\ &\times 2\pi \int d\cos\vartheta_1'' N_{j_1} d^{j_1}_{m_1\lambda'}(\vartheta_1'')\, d^{1/2}_{\nu''\lambda'}(\omega_1'') \\ &\times \Bigg\{ \sum_{j_2 m_2 \mu'} N_{j_2} d^{j_2}_{m_2\mu'}(\vartheta_2'')\, d^{1/2}_{\nu''\mu'}(\omega_2'')\, d^J_{m_1 m_2}(\chi_a'' + \chi_b'')\, e^{-i\pi m_2} \langle m_2 \mu' \mid M_C^{Jj_2} \mid \lambda \rangle \\ &+ \sum_{lm} N_l d^l_{-m0}(\vartheta_3'')\, d^J_{m_1,\nu''-m}(\chi_a'')\, \langle \nu'' m \mid M_C^{Jl} \mid \lambda \rangle \Bigg\}\, D^J_{M\lambda}(r). \end{aligned} \tag{38}$$

When we return to Eq. (30) with results (32) and (38) we see that

$$\begin{aligned} &\mathrm{disc}_{w_1} \langle m_1 \mu \mid M^{Jj_1} \mid \lambda \rangle - 2\pi i\rho_1 \sum_{\lambda'} \langle \mu \mid M^{j_1} \mid \lambda' \rangle_- \langle m_1 \lambda' \mid M^{Jj_1} \mid \lambda \rangle \\ &= 2\pi i \rho_1 \sum_{\lambda'} \langle \mu \mid M^{j_1} \mid \lambda' \rangle_- \cdot 2\pi \sum_{\nu} \int d\cos\vartheta_1 N_{j_1} d^{j_1}_{m_1\lambda'}(\vartheta_1)\, d^{1/2}_{\nu\lambda'}(\omega_1) \\ &\times \Bigg\{ \sum_{j_2 m_2 \mu'} N_{j_2} d^{j_2}_{m_2\mu'}(\vartheta_2)\, d^{1/2}_{\nu\mu'}(\omega_2)\, d^J_{m_1 m_2}(\chi_a + \chi_b)\, e^{-i\pi m_2} \langle m_2 \mu' \mid M_C^{Jj_2} \mid \lambda \rangle \\ &+ \sum_{lm} N_l d^l_{-m0}(\vartheta_3)\, d^J_{m_1,\nu-m}(\chi_a)\, \langle \nu m \mid M_C^{Jl} \mid \lambda \rangle \Bigg\}, \end{aligned} \tag{39}$$

in which double primes have been dropped. This result is a constraint on the amplitude for the production of each πN isobar, as illustrated in Fig. 3. It is a complicated constraint because of the involvement in it of all the KN and $K\pi$ isobar amplitudes. The bothersome phase factor multiplying the KN isobar amplitude is a consequence of the rotation $r_{00\pi}$ in Eq. (26) which arose from our convention about the Euler angles in CM_2 ; it is the price we pay for keeping phases out of Eq. (28) and probably represents the minimum intrusion of such factors. It is the only phase factor which enters throughout and it always enters in the same way: wherever $\langle m_2\mu \mid M^{Jj_2} \mid \lambda\rangle$ appears, $e^{-i\pi m_2}$ appears multiplying it. The πN elastic amplitude $\langle \mu \mid M^{j_1} \mid \lambda\rangle$, in Eqs. (30) on, depends only on w_1 and satisfies the unitarity relation

$$\operatorname{disc}_{w_1} \langle \mu \mid M^{j_1} \mid \lambda\rangle = 2\pi i \rho_1 \sum_{\lambda'} \langle \mu \mid M^{j_1} \mid \lambda'\rangle_- \langle \lambda' \mid M^{j_1} \mid \lambda\rangle. \tag{40}$$

The w_2-discontinuity, Eq. (17) and Fig. (4), is developed in similar fashion to give the constraint on the KN isobar amplitudes:

$$\begin{aligned}
&\operatorname{disc}_{w_2} \langle m_2\mu \mid M^{Jj_2} \mid \lambda\rangle - 2\pi i \rho_2 \sum_{\lambda'} \langle \mu \mid M^{j_2} \mid \lambda'\rangle_- \langle m_2\lambda' \mid M^{Jj_2} \mid \lambda\rangle \\
&= 2\pi i \rho_2 \sum_{\lambda'} \langle \mu \mid M^{j_2} \mid \lambda'\rangle_- \cdot 2\pi e^{i\pi m_2} \sum_{\nu} \int d\cos\vartheta_2 N_{j_2} d^{j_2}_{m_2\lambda'}(\vartheta_2)\, d^{1/2}_{\nu\lambda'}(\omega_2) \\
&\quad \times \Big\{ \sum_{j_1 m_1 \mu'} N_{j_1} d^{j_1}_{m_1\mu'}(\vartheta_1)\, d^{1/2}_{\nu\mu'}(\omega_1)\, d^{J}_{m_1 m_2}(\chi_a + \chi_b) \langle m_1\mu' \mid M^{Jj_1}_{\mathrm{C}} \mid \lambda\rangle \\
&\quad + \sum_{lm} N_l d^{l}_{-m0}(\vartheta_3)\, d^{J}_{\nu-m,m_2}(\chi_b) \langle \nu m \mid M^{Jl}_{\mathrm{C}} \mid \lambda\rangle \Big\}.
\end{aligned} \tag{41}$$

All the πN and $K\pi$ isobar amplitudes appear on the right-hand side. The KN elastic amplitude $\langle \mu \mid M^{j_2} \mid \lambda\rangle$ depends only on w_2 and satisfies

$$\operatorname{disc}_{w_2} \langle \mu \mid M^{j_2} \mid \lambda\rangle = 2\pi i \rho_2 \sum_{\lambda'} \langle \mu \mid M^{j_2} \mid \lambda'\rangle_- \langle \lambda' \mid M^{j_2} \mid \lambda\rangle, \tag{42}$$

where the KN phase space factor is

$$\rho_2(w_2) = M Q_2(w_2)/16\pi^3 w_2 . \tag{43}$$

The development of the x-discontinuity, Eq. (19) and Fig. (5), proceeds along the same lines except for one subtlety which must be incorporated in the initial assembly of the analog to Eq. (30). The integration is over the intermediate $K\pi$ phase space with vectors k'' and p''; in CM_3, k_3'' has Euler angles $(\psi_3''\vartheta_3''0)$. Because the final nucleon (momentum Q, helicity ν) is fixed with Euler angles $(\phi_3\theta_3\psi_3)$ we not only identify r_3'' to have angles $(\phi_3\theta_3\psi_3'')$; we also must include a phase factor $e^{i\nu(\psi_3-\psi_3'')}$. Once this has been allowed for, the procedure leads to the constraint on the $K\pi$ isobar amplitudes:

$$\begin{aligned}
&\operatorname{disc}_x \langle \nu m \mid M^{Jl} \mid \lambda\rangle - 2\pi i \rho_x M_-^{\,l} \langle \nu m \mid M^{Jl} \mid \lambda\rangle \\
&= 2\pi i \rho_x M_-^{\,l} \cdot 2\pi \int d\cos\vartheta_3 N_l d^{l}_{-m0}(\vartheta_3) \\
&\quad \times \Big\{ \sum_{j_1 m_1 \mu} N_{j_1} d^{j_1}_{m_1\mu}(\vartheta_1)\, d^{1/2}_{\nu\mu}(\omega_1)\, d^{J}_{m_1,\nu-m}(\chi_a) \langle m_1\mu \mid M^{Jj_1}_{\mathrm{C}} \mid \lambda\rangle \\
&\quad + \sum_{j_2 m_2 \mu} N_{j_2} d^{j_2}_{m_2\mu}(\vartheta_2)\, d^{1/2}_{\nu\mu}(\omega_2)\, d^{J}_{\nu-m,m_2}(\chi_b)\, e^{-i\pi m_2} \langle m_2\mu \mid M^{Jj_2}_{\mathrm{C}} \mid \lambda\rangle \Big\},
\end{aligned} \tag{44}$$

in which all the πN and KN isobar amplitudes participate. The $K\pi$ elastic amplitude M^l depends only on x and satisfies

$$\text{disc}_x M^l = 2\pi i \rho_x M_-^l M^l \tag{45}$$

with $K\pi$ phase space factor

$$\rho_x(x) = k_3(x)/32\pi^3(x)^{1/2} \tag{46}$$

For reasons which are apparent later, each of results (39), (41) and (44) has been written such that only recoupling terms appear on the right-hand side.

The W-discontinuities are elementary to construct from Eqs. (20). We get

$$\text{disc}_W \langle m_1\mu \mid M^{Jj_1} \mid \lambda\rangle = 2\pi i\rho \sum_{\lambda'} \langle m_1\mu \mid M^{Jj_1} \mid \lambda'\rangle\langle\lambda' \mid M^J \mid \lambda\rangle_- ,$$

similarly for $1 \to 2$, and (47)

$$\text{disc}_W \langle \nu m \mid M^{Jl} \mid \lambda\rangle = 2\pi i\rho \sum_{\lambda'} \langle \nu m \mid M^{Jl} \mid \lambda'\rangle\langle\lambda' \mid M^J \mid \lambda\rangle_- .$$

The last factor is the KN elastic amplitude, a function only of W. It satisfies Eq. (42) with the replacement of J for j_2 and W for w_2 ; thus the phase space factor is the same function of W as ρ_2 is of w_2 :

$$\rho(W) = MP(W)/16\pi^3 W. \tag{48}$$

Since an isobar denotes a state of definite quantum numbers, the isobar amplitudes should be formed with definite isobar parity and the foregoing constraints should be accordingly modified. The parity properties of helicity states are given in [4], and parity conservation is invoked. We only need to take the appropriate combinations of nucleon helicities $+\frac{1}{2}$ and $-\frac{1}{2}$ to do this ($+$ and $-$ for short). The πN and KN elastic amplitudes with definite parity, $p = \pm$, are

$$M^{j^p} = \langle +\mid M^j \mid +\rangle + \eta_p(-)^{j+\frac{1}{2}} \langle -\mid M^j \mid +\rangle \tag{49}$$

in which $\eta_\pm = \pm 1$. The $K\pi$ elastic amplitude M^l of course already has definite parity $(-)^l$. The πN and KN definite parity isobar amplitudes are combinations of $\langle m\mu \mid M^{Jj} \mid \lambda\rangle$ for $\mu = +\frac{1}{2}$ and $-\frac{1}{2}$ (again $+$ and $-$ for short):

$$\langle m \mid M^{Jj^p} \mid \lambda\rangle = (\langle m+\mid M^{Jj} \mid \lambda\rangle + \eta_p(-)^{j+\frac{1}{2}} \langle m-\mid M^{Jj} \mid \lambda\rangle)/2^{1/2}. \tag{50}$$

The $K\pi$ isobar amplitude $\langle \nu m \mid M^{Jl} \mid \lambda\rangle$ already has definite isobar parity $(-)^l$. A very useful construction can be introduced:

$$\sum_{\mu=\pm 1/2} d^j_{m\mu}(\vartheta)\, d^{1/2}_{\nu\mu}(\omega) \langle m\mu \mid M^{Jj} \mid \lambda\rangle = \sum_{p=\pm} e^{j^p}_{m\nu}(\vartheta) \langle m \mid M^{Jj^p} \mid \lambda\rangle, \tag{51}$$

where

$$e^{j^p}_{m\nu}(\vartheta) = (d^j_{m+}(\vartheta)\, d^{1/2}_{\nu+}(\omega) + \eta_p(-)^{j+\frac{1}{2}} d^j_{m-}(\vartheta)\, d^{1/2}_{\nu-}(\omega))/2^{1/2}. \tag{52}$$

By means of Eqs. (49) to (52) we obtain, for amplitudes of definite isobar parity:

$$\begin{aligned}
&\operatorname{disc}_{w_1} \langle m_1 \mid M^{Jj_1^p} \mid \lambda\rangle - 2\pi i\rho_1 M_-^{j_1^p}\langle m_1 \mid M^{Jj_1^p} \mid \lambda\rangle \\
&= 2\pi i\rho_1 M_-^{j_1^p} \cdot 2\pi \sum_\nu \int d\cos\vartheta_1 N_{j_1} e_{m_1\nu}^{j_1^p}(\vartheta_1) \\
&\quad\times \Big\{ \sum_{j_2\pi m_2} N_{j_2} e_{m_2\nu}^{j_2^\pi}(\vartheta_2)\, d^J_{m_1 m_2}(\chi_a + \chi_b)\, e^{-i\pi m_2}\langle m_2 \mid M_C^{Jj_2^\pi} \mid \lambda\rangle \\
&\quad+ \sum_{lm} N_l d^l_{-m0}(\vartheta_3)\, d^J_{m_1,\nu-m}(\chi_a)\, \langle \nu m \mid M_C^{Jl} \mid \lambda\rangle \Big\},
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\operatorname{disc}_{w_2} \langle m_2 \mid M^{Jj_2^p} \mid \lambda\rangle - 2\pi i\rho_2 M_-^{j_2^p}\langle m_2 \mid M^{Jj_2^p} \mid \lambda\rangle \\
&= 2\pi i\rho_2 M_-^{j_2^p} \cdot 2\pi e^{i\pi m_2} \sum_\nu \int d\cos\vartheta_2 N_{j_2} e_{m_2\nu}^{j_2^p}(\vartheta_2) \\
&\quad\times \Big\{ \sum_{j_1\pi m_1} N_{j_1} e_{m_1\nu}^{j_1^\pi}(\vartheta_1)\, d^J_{m_1 m_2}(\chi_a + \chi_b)\, \langle m_1 \mid M_C^{Jj_1^\pi} \mid \lambda\rangle \\
&\quad+ \sum_{lm} N_l d^l_{-m0}(\vartheta_3)\, d^J_{\nu-m,m_2}(\chi_b)\, \langle \nu m \mid M_C^{Jl} \mid \lambda\rangle \Big\},
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&\operatorname{disc}_x \langle \nu m \mid M^{Jl} \mid \lambda\rangle - 2\pi i\rho_x M_-^{l}\langle \nu m \mid M^{Jl} \mid \lambda\rangle \\
&= 2\pi i\rho_x M_-^{l} \cdot 2\pi \int d\cos\vartheta_3 N_l d^l_{-m0}(\vartheta_3) \\
&\quad\times \Big\{ \sum_{j_1\pi m_1} N_{j_1} e_{m_1\nu}^{j_1^\pi}(\vartheta_1)\, d^J_{m_1,\nu-m}(\chi_a)\, \langle m_1 \mid M_C^{Jj_1^\pi} \mid \lambda\rangle \\
&\quad+ \sum_{j_2\pi m_2} N_{j_2} e_{m_2\nu}^{j_2^\pi}(\vartheta_2)\, d^J_{\nu-m,m_2}(\chi_b)\, e^{-i\pi m_2}\langle m_2 \mid M_C^{Jj_2^\pi} \mid \lambda\rangle \Big\}.
\end{aligned} \tag{55}$$

Of course we also have from Eqs. (40) and (42)

$$\operatorname{disc}_{w_1} M^{j_1^p} = 2\pi i\rho_1 M_-^{j_1^p} M^{j_1^p}, \tag{56}$$

and similarly for $1 \to 2$.

Amplitudes with definite overall parity, $P = \pm$, can also be formed in a straightforward way. The discontinuity formulas become lengthy and call for even more notation. We have relegated these results to Appendix B.

5. Isobar Factors

The authors of Refs. [1, 2] have noted the consequences of adopting a certain two-factor form for each term in the isobar expansion. Each isobar amplitude is written as the product of the elastic amplitude for scattering in the isobar state times a residual factor:

$$\langle m_1 \mid M^{Jj_1^{p}} \mid \lambda\rangle = M^{j_1^{p}}\langle m_1 \mid \mathscr{M}^{Jj_1^{p}} \mid \lambda\rangle, \tag{57}$$

similarly for $1 \to 2$, and

$$\langle \nu m \mid M^{Jl} \mid \lambda\rangle = M^{l}\langle \nu m \mid \mathscr{M}^{Jl} \mid \lambda\rangle.$$

We refer to the $\mathscr{M}$'s here as isobar factors.

Expressions (57) are to be incorporated in Eqs. (53), (54), and (55). We then can use the identity

$$\operatorname{disc}(M\mathscr{M}) = (\operatorname{disc} M)\,\mathscr{M} + M_{-}(\operatorname{disc}\mathscr{M}) \tag{58}$$

together with Eqs. (45) and (56) to obtain a cancellation on the left-hand side of the discontinuity formulas. The results are

$$\begin{aligned}
&\operatorname{disc}_{w_1} \langle m_1 \mid \mathscr{M}^{Jj_1^{p}} \mid \lambda\rangle^{Tt_1} \\
&\quad = 2\pi i\rho_1 \cdot 2\pi \sum_{\nu} \int d\cos\vartheta_1 N_{j_1} e^{j_1^{p}}_{m_1\nu}(\vartheta_1) \\
&\qquad \times \Bigg\{ \sum_{\substack{j_2\pi m_2\\ t_2}} N_{j_2} e^{j_2^{\pi}}_{m_2\nu}(\vartheta_2)\, d^{J}_{m_1m_2}(\chi_a + \chi_b)\, e^{-i\pi m_2} C^{T}_{t_1t_2} M^{t_2j_2^{\pi}} \langle m_2 \mid \mathscr{M}^{Jj_2^{\pi}} \mid \lambda\rangle^{Tt_2} \\
&\qquad + \sum_{\substack{lm\\ t}} N_l d^{l}_{-m0}(\vartheta_3)\, d^{J}_{m_1,\nu-m}(\chi_a)\, C^{T}_{t_1t} M^{tl} \langle \nu m \mid \mathscr{M}^{Jl} \mid \lambda\rangle^{Tt} \Bigg\},
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\operatorname{disc}_{w_2} \langle m_2 \mid \mathscr{M}^{Jj_2^{p}} \mid \lambda\rangle^{Tt_2} \\
&\quad = 2\pi i\rho_2 \cdot 2\pi e^{i\pi m_2} \sum_{\nu} \int d\cos\vartheta_2 N_{j_2} e^{j_2^{p}}_{m_2\nu}(\vartheta_2) \\
&\qquad \times \Bigg\{ \sum_{\substack{j_1\pi m_1\\ t_1}} N_{j_1} e^{j_1^{\pi}}_{m_1\nu}(\vartheta_1)\, d^{J}_{m_1m_2}(\chi_a + \chi_b)\, C^{T}_{t_2t_1} M^{t_1j_1^{\pi}} \langle m_1 \mid \mathscr{M}^{Jj_1^{\pi}} \mid \lambda\rangle^{Tt_1} \\
&\qquad + \sum_{\substack{lm\\ t}} N_l d^{l}_{-m0}(\vartheta_3)\, d^{J}_{\nu-m,m_2}(\chi_b)\, C^{T}_{t_2t} M^{tl} \langle \nu m \mid \mathscr{M}^{Jl} \mid \lambda\rangle^{Tt} \Bigg\},
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
&\operatorname{disc}_{x} \langle \nu m \mid \mathscr{M}^{Jl} \mid \lambda\rangle^{Tt} \\
&\quad = 2\pi i\rho_x \cdot 2\pi \int d\cos\vartheta_3 N_l d^{l}_{-m0}(\vartheta_3) \\
&\qquad \times \Bigg\{ \sum_{\substack{j_1\pi m_1\\ t_1}} N_{j_1} e^{j_1^{\pi}}_{m_1\nu}(\vartheta_1)\, d^{J}_{m_1,\nu-m}(\chi_a)\, C^{T}_{tt_1} M^{t_1j_1^{\pi}} \langle m_1 \mid \mathscr{M}^{Jj_1^{\pi}} \mid \lambda\rangle^{Tt_1} \\
&\qquad + \sum_{\substack{j_2\pi m_2\\ t_2}} N_{j_2} e^{j_2^{\pi}}_{m_2\nu}(\vartheta_2)\, d^{J}_{\nu-m,m_2}(\chi_b)\, e^{-i\pi m_2} C^{T}_{tt_2} M^{t_2j_2^{\pi}} \langle m_2 \mid \mathscr{M}^{Jj_2^{\pi}} \mid \lambda\rangle^{Tt_2} \Bigg\}.
\end{aligned} \tag{61}$$

These formulas, in which the isospin dependence has been reinstated, show that the isobar factor discontinuities are determined by the scattering and isobar factors in the other two isobar channels.

6. Dalitz Plot and Other Kinematics

Our discontinuity formulas each involve an integration over the cosine of the polar angle in the isobar rest frame: $\cos\vartheta_1$ for fixed W and w_1, $\cos\vartheta_2$ for fixed W and w_2, and $\cos\vartheta_3$ for fixed W and x. Some kinematical intuition is to be gained by casting these in terms of integrations over the Dalitz plot.

In Figs. 8a–c, we have illustrated the configurations of the vectors Qkp in CM prior to the application of the final rotations r_1, r_2, and r_3; three different rest frames are indicated along with the associated Lorentz transformations: $CM_1 \to CM$ under z_1, $CM_2 \to CM$ under z_2, and $CM_3 \to CM$ under z_3. The isobar vectors Q_a, Q_b, and K_c are transformed from rest by z_1, z_2, and z_3, respectively. From $Q = z_1 Q_1$ we deduce

$$\cos\vartheta_1 = (Q_0 w_1 - Q_{a0} Q_{10})/Q_a Q_1 . \tag{62}$$

From $Q = z_2 Q_2$ we obtain

$$\cos\vartheta_2 = (Q_0 w_2 - Q_{b0} Q_{20})/Q_b Q_2 . \tag{63}$$

From $k = z_3 k_3$ we get

$$\cos\vartheta_3 = -(k_0 x^{1/2} - K_{c0} k_{30})/Q k_3 . \tag{64}$$

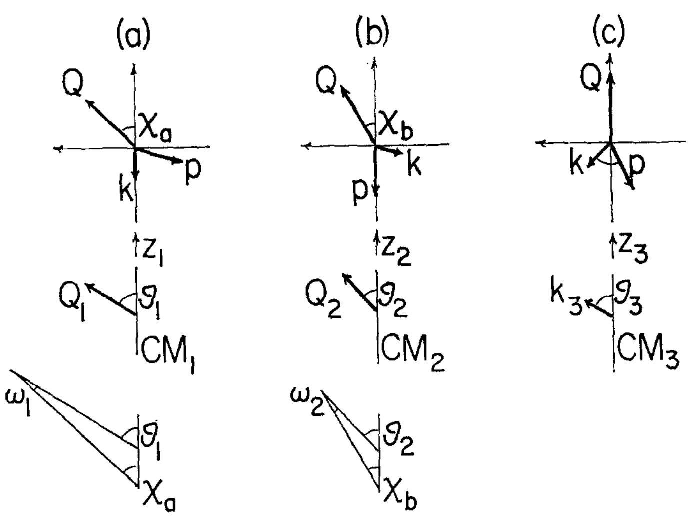

Fig. 8. Three orientations of the vectors $Q\ k\ p$ in CM in which: (a) Q_a is along the Z axis, (b) Q_b is along the Z axis, and (c) K_c is along the $-Z$ axis. The rest frames of the isobars Q_a, Q_b, and K_c are shown. The non-Euclidean figures for the determination of the spin-rotation angles ω_1 and ω_2 are also shown.

In terms of the invariants, the energies and momenta are

$$Q_0 = (W^2 + M^2 - x)/2W = (w_1^2 + w_2^2 - m^2 - \mu^2)/2W$$
$$\text{and} \quad Q = (Q_0^2 - M^2)^{1/2}, \tag{65}$$

$$k_0 = (W^2 + m^2 - w_1^2)/2W = (w_2^2 + x - M^2 - \mu^2)/2W, \tag{66}$$

$$Q_{a0} = (W^2 + w_1^2 - m^2)/2W \quad \text{and} \quad Q_a = (Q_{a0}^2 - w_1^2)^{1/2}, \tag{67}$$

$$Q_{b0} = (W^2 + w_2^2 - \mu^2)/2W \quad \text{and} \quad Q_b = (Q_{b0}^2 - w_2^2)^{1/2}, \tag{68}$$

$$K_{c0} = (W^2 + x - M^2)/2W, \tag{69}$$

$$Q_{10} = (w_1^2 + M^2 - \mu^2)/2w_1 \quad \text{and} \quad Q_1 = (Q_{10}^2 - M^2)^{1/2}, \tag{70}$$

$$Q_{20} = (w_2^2 + M^2 - m^2)/2w_2 \quad \text{and} \quad Q_2 = (Q_{20}^2 - M^2)^{1/2}, \tag{71}$$

and

$$k_{30} = (x + m^2 - \mu^2)/2x^{1/2} \quad \text{and} \quad k_3 = (k_{30}^2 - m^2)^{1/2}. \tag{72}$$

In Eqs. (59), (60), and (61) we want to transform the integration variables such that the πN, KN, and $K\pi$ isobar contributions are integrated over w_1, w_2, and x, respectively. For the $\cos \vartheta_1$ integration at fixed W and w_1, we use (62) to get

$$d \cos \vartheta_1 = (w_1 w_2 / W Q_a Q_1)\, dw_2 = -(w_1/2WQ_aQ_1)\, dx; \tag{73}$$

for the $\cos \vartheta_2$ integration at fixed W and w_2 we use (63) to get

$$d \cos \vartheta_2 = (w_1 w_2 / W Q_b Q_2)\, dw_1 = -(w_2/2WQ_bQ_2)\, dx; \tag{74}$$

for the $\cos \vartheta_3$ integration at fixed W and x we use (64) to get

$$d \cos \vartheta_3 = (w_1 x^{1/2} / WQk_3)\, dw_1 = -(w_2 x^{1/2}/WQk_3)\, dw_2\,. \tag{75}$$

The integrations are over traversals of the Dalitz plot as shown in Fig. 9.

All the angles appearing in the integrands can be related to the Dalitz plot variables. In Fig. 8a the angle χ_a in CM is identified. When we consider that $Q = z_1 Q_1$ and use Eq. (62) we find that

$$\cos \chi_a = (Q_0 Q_{a0} - w_1 Q_{10})/Q_a Q. \tag{76}$$

Likewise, Fig. 8b shows the angle χ_b in CM so that when we use the relation $Q = z_2 Q_2$ together with Eq. (63) we get

$$\cos \chi_b = (Q_0 Q_{b0} - w_2 Q_{20})/Q_b Q. \tag{77}$$

The angles ω_1 and ω_2 remain to be identified; these describe the rotation of the nucleon spin in passing, respectively, from CM_1 to CM and from CM_2 to CM. If we consult

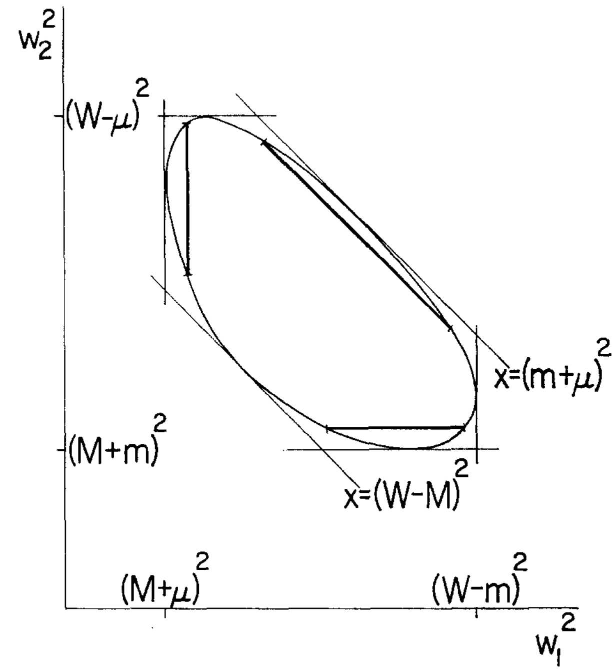

FIG. 9. The Dalitz plot for given W. Integrations at fixed w_1, w_2, and x are indicated.

[3, Appendix] and compare with the non-Euclidean triangles drawn in Figs. 8a and b we conclude that

$$\cos \omega_1 = (Q_0 Q_{10} - M^2 Q_{a0}/w_1)/QQ_1, \tag{78}$$

$$(w_1/Q_a) \sin \omega_1 = (M/Q) \sin \vartheta_1 = (M/Q_1) \sin \chi_a, \tag{79}$$

$$\cos \omega_2 = (Q_0 Q_{20} - M^2 Q_{b0}/w_2)/QQ_2, \tag{80}$$

and

$$(w_2/Q_b) \sin \omega_2 = (M/Q) \sin \vartheta_2 = (M/Q_2) \sin \chi_b. \tag{81}$$

All of the cosine formulas in this section are rather lengthy functions of the invariants, owing to the unequal mass kinematics.

7. Application

A concluding example serves to illustrate our constraints put to use. We choose a set of circumstances for which each of the angular momentum summations can legitimately be truncated to a single dominating term.

We consider $KN \to K\pi N$ at energies only a little larger than threshold. Because of centrifugal barrier effects it is then reasonable to suppose that only the s-wave systems in the final state will have appreciable amplitudes. Therefore we can discard all but one of the isobar amplitudes in each isobar channel. We keep only the πN

and KN isobars having $j^p = \frac{1}{2}^-$, and we keep only the $l = 0$ $K\pi$ isobar. The restriction to s-waves in all aspects of the final state means that only $J^P = \frac{1}{2}^+$ is of interest.

If we address ourselves to Eqs. (53), (54), and (55) for $J^P = \frac{1}{2}^+$ we see from Eqs. (B1) and (B2) that we wish to form combinations for which $\kappa_1 = \kappa_2 = \frac{1}{2}$ (denoted $+$, for short) and $\xi = 0$; thus we have

$$M^{\frac{1}{2}^+ \frac{1}{2}^- +} = \langle + | \, M^{\frac{1}{2} \frac{1}{2}^-} | + \rangle - \langle + | \, M^{\frac{1}{2} \frac{1}{2}^-} | - \rangle \tag{82}$$

for both the πN and KN isobars, and

$$M^{\frac{1}{2}^+ 0\, 0} = \langle +0 \,| \, M^{\frac{1}{2}\, 0} | + \rangle - \langle +0 \,| \, M^{\frac{1}{2}\, 0} | - \rangle \tag{83}$$

for the $K\pi$ isobar. The formulas in Appendix B are most convenient because they provide for such combinations directly.

At low energy the effect of nucleon spin rotation is negligible. As the spin rotation angle $\omega \to 0$, $d^{1/2}_{\mu\nu}(\omega) \to \delta_{\mu\nu}$, so that $e^{1/2^-}_{\mu\nu}(\vartheta) \to d^{1/2}_{\mu\nu}(\vartheta)/2^{1/2}$ in Eq. (52). Calculations ($B8$) and ($B9$), with a table of d-functions, then give:

$$f^{\frac{1}{2}^- \frac{1}{2}^-}_{+\ +}(\vartheta\vartheta') = \tfrac{1}{2} \cos(\vartheta - \vartheta')/2, \qquad f^{\frac{1}{2}^- \frac{1}{2}^-}_{+\ -}(\vartheta\vartheta') = -\tfrac{1}{2} \sin(\vartheta - \vartheta')/2$$
$$f^{\frac{1}{2}^-\, 0+}_{+\ \ 0}(\vartheta\vartheta_3) = \tfrac{1}{2} \cos(\vartheta/2), \qquad f^{\frac{1}{2}^-\, 0-}_{+\ \ 0}(\vartheta\vartheta_3) = -\tfrac{1}{2} \sin(\vartheta/2) \tag{84}$$

and

$$f^{\frac{1}{2}^-\, 0+}_{-\ \ 0}(\vartheta\vartheta_3) = \tfrac{1}{2} \sin(\vartheta/2).$$

We adopt an abbreviated notation for the amplitudes since we need only work with one of each: we call the πN and KN isobar amplitudes in (82) $M(Ww_1)$ and $M(Ww_2)$, and the $K\pi$ isobar amplitude in (83) $M(Wx)$. The elastic amplitudes for πN, KN, and $K\pi$ we denote $M(w_1)$, $M(w_2)$, and $M(x)$. Equations (B5), (B6), and (B7) become

$$\begin{aligned} &\mathrm{disc}_{w_1} M(Ww_1) - 2\pi i \rho_1 M_-(w_1)\, M(Ww_1) \\ &\quad = 2\pi i \rho_1 M_-(w_1) \int \frac{d\cos\vartheta_1}{2} \Big\{ \cos \frac{(\vartheta_1 + \chi_a) - (\vartheta_2 - \chi_b)}{2}\, e^{-i\pi/2} M_C(Ww_2) \\ &\qquad + \cos \frac{\vartheta_1 - \chi_a}{2}\, M_C(Wx) \Big\}, \end{aligned} \tag{85}$$

$$\begin{aligned} &\mathrm{disc}_{w_2} M(Ww_2) - 2\pi i \rho_2 M_-(w_2)\, M(Ww_2) \\ &\quad = 2\pi i \rho_2 e^{i\pi/2} M_-(w_2) \int \frac{d\cos\vartheta_2}{2} \Big\{ \cos \frac{(\vartheta_1 - \chi_a) - (\vartheta_2 + \chi_b)}{2}\, M_C(Ww_1) \\ &\qquad + \cos \frac{\vartheta_2 + \chi_b}{2}\, M_C(Wx) \Big\}, \end{aligned} \tag{86}$$

and

$$\begin{aligned} &\mathrm{disc}_x M(Wx) - 2\pi i \rho_x M_-(x)\, M(Wx) \\ &\quad = 2\pi i \rho_x M_-(x) \int \frac{d\cos\vartheta_3}{2} \Big\{ \cos \frac{\vartheta_1 - \chi_a}{2}\, M_C(Ww_1) \\ &\qquad + \cos \frac{\vartheta_2 - \chi_b}{2}\, e^{-i\pi/2} M_C(Ww_2) \Big\}. \end{aligned} \tag{87}$$

These equations form a coupled system of constraints on the set of isobar amplitudes, in principle the basis for further investigation, either dynamical or phenomenological.

APPENDIX A. ISOSPIN PROJECTION OPERATORS

The isospin projection operators used in Section 2 are listed here. For the elastic amplitudes they are:

$a_{ji}^{t_1}$ for $\pi_i N \to \pi_j N$ ($t_1 = \frac{1}{2}$ and $\frac{3}{2}$):

$$\begin{aligned} a_{ji}^{1/2} &= (\delta_{ji} + i\epsilon_{jik}\tau_k{}^N)/3, \\ a_{ji}^{3/2} &= (2\delta_{ji} - i\epsilon_{jik}\tau_k{}^N)/3; \end{aligned} \tag{A1}$$

b^{t_2} for $KN \to KN$ ($t_2 = 0$ and 1):

$$\begin{aligned} b^0 &= (1 - \boldsymbol{\tau}^K \cdot \boldsymbol{\tau}^N)/4, \\ b^1 &= (3 + \boldsymbol{\tau}^K \cdot \boldsymbol{\tau}^N)/4; \end{aligned} \tag{A2}$$

c_{ji}^t for $K\pi_i \to K\pi_j$ ($t = \frac{1}{2}$ and $\frac{3}{2}$):

$$\begin{aligned} c_{ji}^{1/2} &= (\delta_{ji} + i\epsilon_{jik}\tau_k{}^K)/3, \\ c_{ji}^{3/2} &= (2\delta_{ji} - i\epsilon_{jik}\tau_k{}^K)/3. \end{aligned} \tag{A3}$$

For the isobar amplitudes they are:

$\mathscr{A}_i^{Tt_1}$ for$KN \to (\pi_i N)\, K$ ($T = 0$, $t_1 = \frac{1}{2}$; $T = 1$, $t_1 = \frac{1}{2}$ and $\frac{3}{2}$):

$$\begin{aligned} \mathscr{A}_i^{0\,\frac{1}{2}} &= -(1/4\,(3^{1/2}))(\boldsymbol{\tau}^K - \boldsymbol{\tau}^N + i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{A}_i^{1\,\frac{1}{2}} &= -(1/4\,(3^{1/2}))(3\boldsymbol{\tau}^N + \boldsymbol{\tau}^K + i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{A}_i^{1\,\frac{3}{2}} &= -(1/2\,(6^{1/2}))(2\boldsymbol{\tau}^K - i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,; \end{aligned} \tag{A4}$$

$\mathscr{B}_i^{Tt_2}$ for $KN \to (KN)\pi_i$ ($T = 0$, $t_2 = 1$; $T = 1$, $t_2 = 0$ and 1):

$$\begin{aligned} \mathscr{B}_i^{01} &= -(1/4\,(3^{1/2}))(\boldsymbol{\tau}^K - \boldsymbol{\tau}^N + i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{B}_i^{10} &= \tfrac{1}{4}(\boldsymbol{\tau}^K - \boldsymbol{\tau}^N - i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{B}_i^{11} &= (1/2(2^{1/2}))(\boldsymbol{\tau}^K + \boldsymbol{\tau}^N)_i\,; \end{aligned} \tag{A5}$$

$\mathscr{C}_i^{Tt}$ for $KN \to (K\pi_i)\, N$ ($T = 0$, $t = \frac{1}{2}$; $T = 1$, $t = \frac{1}{2}$ and $\frac{3}{2}$):

$$\begin{aligned} \mathscr{C}_i^{0\,\frac{1}{2}} &= -(1/4(3^{1/2}))(\boldsymbol{\tau}^K - \boldsymbol{\tau}^N + i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{C}_i^{1\,\frac{1}{2}} &= -(1/4(3^{1/2}))(3\boldsymbol{\tau}^K + \boldsymbol{\tau}^N - i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,, \\ \mathscr{C}_i^{1\,\frac{3}{2}} &= -(1/2(6^{1/2}))(2\boldsymbol{\tau}^N + i\boldsymbol{\tau}^K \times \boldsymbol{\tau}^N)_i\,. \end{aligned} \tag{A6}$$

In Section 2 we need formulas for products of (A1) with (A4) to (A6), (A2) with (A4) to (A6), and (A3) with (A4) to (A6). It can be shown that

$$\begin{aligned}
\sum_i a_{ji}^{t_1'} \mathcal{A}_i^{Tt_1} &= \mathcal{A}_j^{Tt_1} \delta_{t_1 t_1'}\,, \\
\sum_i a_{ji}^{t_1} \mathcal{B}_i^{Tt_2} &= \mathcal{A}_j^{Tt_1} C_{t_1 t_2}^{T}\,, \\
\sum_i a_{ji}^{t_1} \mathcal{C}_i^{Tt} &= \mathcal{A}_j^{Tt_1} C_{t_1 t}^{T}\,;
\end{aligned} \tag{A7}$$

$$\begin{aligned}
\ell^{t_2} \mathcal{A}_j^{Tt_1} &= \mathcal{B}_j^{Tt_2} C_{t_2 t_1}^{T}\,, \\
\ell^{t_2'} \mathcal{B}_j^{Tt_2} &= \mathcal{B}_j^{Tt_2} \delta_{t_2 t_2'}\,, \\
\ell^{t_2} \mathcal{C}_j^{Tt} &= \mathcal{B}_j^{Tt_2} C_{t_2 t}^{T}\,;
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\sum_i c_{ji}^{t} \mathcal{A}_i^{Tt_1} &= \mathcal{C}_j^{Tt} C_{tt_1}^{T}\,, \\
\sum_i c_{ji}^{t} \mathcal{B}_i^{Tt_2} &= \mathcal{C}_j^{Tt} C_{tt_2}^{T}\,, \\
\sum_i c_{ji}^{t'} \mathcal{C}_i^{Tt} &= \mathcal{C}_j^{Tt} \delta_{tt'}\,.
\end{aligned} \tag{A9}$$

The C's appearing in (A7) to (A9) are constants which have been recorded in the text in Table I.

APPENDIX B. Isobar Amplitudes of Definite Parity

For the production of πN or KN isobars, of spin-parity j^p, there are $j + \frac{1}{2}$ independent amplitudes of definite angular momentum J and definite parity P. We label these as

$$\kappa = j, \ldots, \tfrac{1}{2} \qquad (j + \tfrac{1}{2} \text{ values}).$$

The amplitudes are combinations of $\langle m \mid M^{Jj^p} \mid \lambda \rangle$:

$$\begin{aligned}
M^{J^P j^p \kappa} &= \langle \kappa \mid M^{Jj^p} \mid + \rangle + \eta_P \eta_p (-)^{J+j} \langle -\kappa \mid M^{Jj^p} \mid + \rangle \\
&= \langle \kappa \mid M^{Jj^p} \mid + \rangle + \eta_P (-)^{J+\frac{1}{2}} \langle \kappa \mid M^{Jj^p} \mid - \rangle .
\end{aligned} \tag{B1}$$

For the production of $K\pi$ isobars, of spin l and parity $(-)^l$, there are $2l + 1$ independent amplitudes of definite J^P. We label these as

$$\xi = l, \ldots, -l \qquad (2l + 1 \text{ values}).$$

The amplitudes are combinations of $\langle \nu m \mid M^{Jl} \mid \lambda \rangle$:

$$
\begin{aligned}
M^{J^P l \xi} &= \langle +\xi \mid M^{Jl} \mid + \rangle + \eta_P (-)^{J-\frac{1}{2}} \langle - \, -\xi \mid M^{Jl} \mid + \rangle \\
&= \langle +\xi \mid M^{Jl} \mid + \rangle + \eta_P (-)^{J+\frac{1}{2}} \langle +\xi \mid M^{Jl} \mid - \rangle .
\end{aligned} \tag{B2}
$$

When (B1) and (B2) are employed, and the appropriate combinations are taken, Eqs. (47) become

$$\operatorname{disc}_W M^{J^P j_1{}^p \kappa_1} = 2\pi i \rho M^{J^P j_1{}^p \kappa_1} M_-^{J^P},$$

similarly for $1 \to 2$, and

$$\operatorname{disc}_W M^{J^P l \xi} = 2\pi i \rho M^{J^P l \xi} M_-^{J^P}. \tag{B4}$$

The lengthier subenergy discontinuity formulas are obtained from (53), (54), and (55), using (B1) and (B2):

$$
\begin{aligned}
&\operatorname{disc}_{w_1} M^{J^P j_1{}^p \kappa_1} - 2\pi i \rho_1 M_-^{j_1{}^p} M^{J^P j_1{}^p \kappa_1} \\
&\quad = 2\pi i \rho_1 M_-^{j_1{}^p} \int d\cos\vartheta_1 \Big\{ \sum_{j_2 \pi \kappa_2} \big(f_{\kappa_1 \kappa_2}^{j_1{}^p j_2{}^\pi}(\vartheta_1 \vartheta_2)\, d^J_{\kappa_1 \kappa_2}(\chi_a + \chi_b) \\
&\qquad - \eta_P \eta_\pi (-)^{J+j_2} f_{\kappa_1 -\kappa_2}^{j_1{}^p j_2{}^\pi}(\vartheta_1 \vartheta_2)\, d^J_{\kappa_1 -\kappa_2}(\chi_a + \chi_b) \big)\, e^{-i\pi\kappa_2} M_{\mathrm{C}}^{J^P j_2{}^\pi \kappa_2} \\
&\qquad + \sum_{l\xi} \big(f_{\kappa_1 \xi}^{j_1{}^p l+}(\vartheta_1 \vartheta_3)\, d^J_{\kappa_1, \frac{1}{2}-\xi}(\chi_a) + \eta_P (-)^{J-\frac{1}{2}} f_{\kappa_1 \xi}^{j_1{}^p l-}(\vartheta_1 \vartheta_3)\, d^J_{\kappa_1, -\frac{1}{2}+\xi}(\chi_a) \big) M_{\mathrm{C}}^{J^P l \xi} \Big\},
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
&\operatorname{disc}_{w_2} M^{J^P j_2{}^p \kappa_2} - 2\pi i \rho_2 M_-^{j_2{}^p} M^{J^P j_2{}^p \kappa_2} \\
&\quad = 2\pi i \rho_2 e^{i\pi\kappa_2} M_-^{j_2{}^p} \int d\cos\vartheta_2 \Big\{ \sum_{j_1 \pi \kappa_1} \big(f_{\kappa_2 \kappa_1}^{j_2{}^p j_1{}^\pi}(\vartheta_2 \vartheta_1)\, d^J_{\kappa_1 \kappa_2}(\chi_a + \chi_b) \\
&\qquad + \eta_P \eta_\pi (-)^{J+j_1} f_{\kappa_2 -\kappa_1}^{j_2{}^p j_1{}^\pi}(\vartheta_2 \vartheta_1)\, d^J_{-\kappa_1 \kappa_2}(\chi_a + \chi_b) \big) M_{\mathrm{C}}^{J^P j_1{}^\pi \kappa_1} \\
&\qquad + \sum_{l\xi} \big(f_{\kappa_2 \xi}^{j_2{}^p l+}(\vartheta_2 \vartheta_3)\, d^J_{\frac{1}{2}-\xi, \kappa_2}(\chi_b) + \eta_P (-)^{J-\frac{1}{2}} f_{\kappa_2 \xi}^{j_2{}^p l-}(\vartheta_2 \vartheta_3)\, d^J_{-\frac{1}{2}+\xi, \kappa_2}(\chi_b) \big) M_{\mathrm{C}}^{J^P l \xi} \Big\}.
\end{aligned} \tag{B6}
$$

and

$$
\begin{aligned}
&\operatorname{disc}_x M^{J^P l \xi} - 2\pi i \rho_x M_-{}^l M^{J^P l \xi} \\
&\quad = 2\pi i \rho_x M_-{}^l \int d\cos\vartheta_3 \Big\{ \sum_{j_1 \pi \kappa_1} \big(f_{\kappa_1 \xi}^{j_1{}^\pi l+}(\vartheta_1 \vartheta_3)\, d^J_{\kappa_1, \frac{1}{2}-\xi}(\chi_a) \\
&\qquad + \eta_P \eta_\pi (-)^{J+j_1} f_{-\kappa_1 \xi}^{j_1{}^\pi l+}(\vartheta_1 \vartheta_3)\, d^J_{-\kappa_1, \frac{1}{2}-\xi}(\chi_a) \big) M_{\mathrm{C}}^{J^P j_1{}^\pi \kappa_1} \\
&\qquad + \sum_{j_2 \pi \kappa_2} \big(f_{\kappa_2 \xi}^{j_2{}^\pi l+}(\vartheta_2 \vartheta_3)\, d^J_{\frac{1}{2}-\xi, \kappa_2}(\chi_b) \\
&\qquad - \eta_P \eta_\pi (-)^{J+j_2} f_{-\kappa_2 \xi}^{j_2{}^\pi l+}(\vartheta_2 \vartheta_3)\, d^J_{\frac{1}{2}-\xi, -\kappa_2}(\chi_b) \big)\, e^{-i\pi\kappa_2} M_{\mathrm{C}}^{J^P j_2{}^\pi \kappa_2} \Big\}.
\end{aligned} \tag{B7}
$$

In these formulas we have introduced the notation

$$f_{\kappa\kappa'}^{j^{\mathfrak{p}}j'^{\pi}}(\vartheta\vartheta') = 2\pi N_j N_{j'} \sum_{\mu=\pm 1/2} e_{\kappa\mu}^{j^{\mathfrak{p}}}(\vartheta)\, e_{\kappa'\mu}^{j'^{\pi}}(\vartheta') \tag{B8}$$

and

$$f_{\kappa\xi}^{j^{\mathfrak{p}}l\pm}(\vartheta\vartheta_3) = 2\pi N_j N_l e_{\kappa\pm}^{j^{\mathfrak{p}}}(\vartheta)\, d_{\mp\xi 0}^{l}(\vartheta_3). \tag{B9}$$

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